The Seifert conjecture and groups which are coarse quasiisometric to planes
by
Geoffrey Mess
University of California at Los Angeles

Let M be a closed P^2-irreducible 3-manifold such that $\pi_1 M$ has an infinite cyclic normal subgroup $\mathbf{Z}$. This paper proves that $\pi_1 M/\mathbf{Z}$ has the geometry of the euclidean or the hyperbolic plane, in Gromov's coarse quasiisometric sense. This is a step towards the Seifert conjecture, that is, that M satisfying the given conditions must be a Seifert fiber space.

The abelian groups which arise as fundamental groups of 3-manifolds are $\mathbf{Z}$, $\mathbf{Z}\oplus\mathbf{Z}$, $\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$, arbitrary infinitely generated subgroups of the rationals $\mathbf{Q}$, finite cyclic groups, and $\mathbf{Z}_2 \oplus \mathbf{Z}$([H]). It has been conjectured that infinitely generated groups of rank 1 do not occur as normal subgroups of finitely generated 3-manifold groups. This is proved in §2. We review previous related work. Given a compact connected 3-manifold M, with no 2-spheres in the boundary ∂M and no fake cells in M, if $\pi_1 M$ contains an abelian normal subgroup then a) M is an S^2 bundle over S^1 or b) M is homotopy equivalent to $\mathbf{R}P^2 \times S^1$ or c) $\pi_1 M$ is finite or d) $M = \mathbf{R}P^3 \# \mathbf{R}P^3$ up to homotopy or e) M is aspherical or f) capping boundary 2-spheres in the orientation cover of M yields an aspherical 3-manifold. Furthermore, all connected 3-manifolds M such that $\pi_1 M$ contains $\mathbf{Z}\oplus\mathbf{Z}$ or $\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$ as a normal subgroup are known [HJ], [MS]. In addition, recall [EJ] that if an infinitely generated rank 1 abelian subgroup A of a 3-manifold group G satisfies $N_G(A) \neq Z_G(A)$, then $N_G(A) = \mathbf{Z}_2 \ltimes Z_G(A)$, and either $Z_G(A)$ is abelian or $Z_G(A)$ is the fundamental group of a closed aspherical 3-manifold. Passing if need be to a double cover, we need only consider compact aspherical manifolds M, with $Z(\pi_1 M)$ a rank 1 abelian group.

Recall that if M is a Haken manifold and $\pi_1 M$ contains a cyclic normal subgroup, then M is a Seifert fibered space [W],[GH],[JS], and that if M is virtually Haken then by [K] M has the homotopy type and by [S1] the homeomorphism type of a Seifert fibered space. A Haken (or virtually Haken) manifold can contain no infinitely generated subgroup of $\mathbf{Q}$ by [EJ] and roots and centralizers are well understood by [Sh],[JS]. In particular there are no infinitely divisible elements.

Without the hypothesis that M is virtually Haken it is known that if $\pi_1 M$ contains a subgroup $Z\oplus Z$, then either M contains an embedded incompressible torus, and therefore a characteristic manifold [S2, S3] or $\pi_1 M$ has a cyclic normal subgroup. If $Z(\pi_1 M)$ is infinitely generated, then $\pi_1 M/\langle a\rangle$, where a is any nontrivial element of $Z(\pi_1 M)$, is a finitely presented torsion group [EJ] by the classification of abelian 3-manifold groups. As an exercise, the reader may show that if $\mathbf{Q}$ occurs as $Z(\pi_1 M)$ for some 3-manifold M, then $\pi_1 M \simeq \mathbf{Q}$. This is an unpublished observation of Scott. Let $\langle a\rangle$ be an infinite cyclic

central subgroup. In §3 and §4 it is shown that $\pi_1 M/\langle a\rangle$ is coarse quasiisometric to either the euclidean or the hyperbolic plane. In either case it is not torsion. This is a converse to the Torus Theorem of [S2] and together with work of Tukia [Tu1] and Scott [S1] provides a partial solution to the Seifert conjecture, that is, that if M is P^2-irreducible M is a Seifert fiber space.

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Notation. If H is a subgroup of a group G, $N_G(H)$ and $Z_G(H)$ are the normalizer and centralizer of H in G. $Z(G)$ is the center of G. In a metric space, $N(C,X) = \{x : d(x,X) \le C\}$ for a constant C and subset X. X^0 = interior of X.

§*1. The covering corresponding to the center.*

The notation $\hat{M}$ introduced in the statement of theorem 1 will be retained throughout the paper.

THEOREM 1. *Let M be a closed P^2-irreducible 3-manifold. Suppose that $Z(\pi_1 M)$ has rank 1. The covering space $(\hat{M}, P : \hat{M} \to M)$ such that $\pi_1(P)(\pi_1\hat{M}) = Z(\pi_1 M)$ is homeomorphic to $S^1 \times \mathbf{R}^2$.*

PROOF. There is a homotopy $H : M \times [0,1] \to M$ such that $H|M \times \{0\} = \text{id}$, $H|M \times \{1\} = \text{id}$, and each track $H_p : S^1 \to M$ defined by $S^1 = [0,1]/(\{0\} = \{1\})$, $H_p(t) = H(p,t)$ represents $a \in Z(\pi_1(M,p))$. (Because a is central, a is well defined as an element of $\pi_1(M,p)$ for any p.) Since M is aspherical, H can be constructed by extension over the skeleta of M. We consider H as a cyclic homotopy $H : M \times S^1 \to M$. H lifts to a homotopy $\hat{H} : \hat{M} \times S^1 \to \hat{M}$ such that $\hat{H}|_{\hat{M}\times\{0\}} = \text{id}$. Fixing a Riemannian metric (or triangulation) in M and pulling it back to $\hat{M}$, all tracks of $\hat{H}$ in $\hat{M}$ have diameter (measured in the metric, or by the number of simplices required to cover the track) bounded by some constant C, because M is compact.

LEMMA 1. *$\hat{M}$ is oriented.*

PROOF. In other words we must show that each deck transformation of the universal cover of $\hat{M}$ is orientation preserving. But each deck transformation (for example the one corresponding to a) is properly homotopic to the identity (e.g. by a lift of $\hat{H}$ to the universal cover) and therefore is the identity on third homology with locally finite chains. □

LEMMA 2. *$\hat{M}$ has one end (which will be referred to as ∞).*

After passing to the orientation cover of M if necessary, $H_3(M,\mathbf{Z}) = \mathbf{Z}$, while $H_3(Z(\pi_1 M),\mathbf{Z}) = 0$, so $\hat{M}$ is noncompact. Since M is aspherical, $H_2(\hat{M},\mathbf{Z}) = H_2(Z(\pi_1 M),\mathbf{Z}) = 0$, so $\hat{M}$ has only one end. □

LEMMA 3. *Given a compact set K in $\hat{M}$ and $a \in Z(\pi_1 M)$ there exists a torus T separating K from ∞. Moreover, letting E be the noncompact component of $\hat{M}$ split on T, $i : E \to \hat{M}$ the inclusion, there exists A in $Z(\pi_1 E)$ such that $i_* A = a$.*

PROOF. Let K be a compact set in $\hat{M}$. Enlarging K, we can assume that K is not separated from ∞ in $\hat{M}$ by any 2-sphere. (This condition is satisfied as soon as $\pi_1 K \to \pi_1 \hat{M}$ is nontrivial.) In addition assume that $\hat{M} - K$ has only one component.

Then $M-K$ is aspherical: every embedded sphere in $\hat{M}$ bounds a homotopy ball (in fact a ball by [MSY], [D]) so by the sphere theorem, if $\hat{M}$ is not aspherical there is a 2-sphere separating K from ∞ in $\hat{M}$. Let K', K'' be compact codimension 0 submanifolds of $\hat{M}$, such that $K \subset K' \subset K''$, $\partial K'$ and $\partial K''$ are connected, and $d(\partial K, \partial K'), d(\partial K', \partial K'') > C$. Then $\hat{H}(\partial K'' \times S^1)$ is disjoint from $\hat{M}-K'$. The fundamental class of the surface $\partial K''$ is a nontrivial element (in fact a generator) of $H_2(\hat{M}-K', \mathbf{Z})$. Choose a basepoint $*$ for $\hat{M}-K'^0$ on $\partial K''$. Since the image in $\pi_1\hat{M}$ of $(\pi_1\hat{H})(\pi_1(\{*\}\times S^1))$ is the subgroup $\langle a\rangle$ of $\pi_1\hat{M}$, $(\pi_1\hat{H})(\pi_1(\partial K''\times S^1)) \subseteq \pi_1(\hat{M}-K')$ is a finitely generated subgroup with nontrivial center (which contains $(\pi_1\hat{H})(\pi_1(\{*\}\times S^1))$). Since $\hat{M}-K'$ contains no closed 3-manifold as a prime factor, $(\pi_1\hat{H})(\pi_1(\partial K''\times S^1))$, which we denote by G, is the fundamental group of a Seifert fibered manifold with nonempty boundary by the result of [W] applied to a core of the covering space $\hat{M}_G$ determined by G. The inclusion of $\partial K'' \hookrightarrow \hat{M}-K'$ factors through the covering space $\hat{M}_G$ of $\hat{M}-K'$. Since $\hat{M}-K'$ is aspherical, every element of $H_2(\hat{M}_G, \mathbf{Z})$ is represented by an element of $H_2(G, \mathbf{Z})$. G is an extension $\mathbf{Z} \rightarrowtail G \twoheadrightarrow *_{i=1}^{r}\mathbf{Z} * *_{i=1}^{k}\mathbf{Z}_{n_i}$, so every element of $H_2(G,\mathbf{Z})$ is represented by a torus subgroup $\mathbf{Z}\oplus\mathbf{Z} \subset G$. So there is a π_1-injective map $f: T^2 \to \hat{M}-K'$ such that $f_*[T^2]$ is a generator of $H_2(\hat{M}-K', \mathbf{Z})$. It follows that if K''' is a compact submanifold of $\hat{M}$, with connected boundary, and large enough to contain $\hat{H}(K''\times S^1)$ in its interior, then the characteristic submanifold of $K'''-K'^0$ is nonempty and some embedded torus T in the characteristic submanifold separates K' from $\partial K'''$ and therefore from ∞. T is incompressible in $\hat{M}-K$ because K lies inside no ball. Let $E=\hat{M}-S^0$ where S is the compact manifold with boundary T. T is compressible in S, so using [MSY],[D] T bounds a solid torus. Choose a basepoint $*$ in E–perhaps farther toward ∞ than the previous $*$—such that $\hat{H}(\{*\}\times S^1)$ lies in E. Let A be the element of $\pi_1(E,*)$ represented by $\hat{H}(\{*\}\times S^1)$. Then $j_*A \in \pi_1(\hat{M}-K,*)$ centralizes the subgroup $\pi_1(E,*)$ of $\pi_1(\hat{M}-K,*)$. So $A\in Z(\pi_1 E)$, and $i_*A = a$. □

Now we show that $\hat{M}$ has a standard end $T^2\times[0,\infty)$. Fix T, S as above. Given a compact set $L\supset S$, there is a torus T' separating L from ∞, and such that the compact manifold D with boundary components T and T' contains $*$ and $\hat{H}(\{*\}\times S^1)$. T' is incompressible in E because S lies in no ball. So $\pi_1(D,*)$ is a subgroup of $\pi_1(\hat{M}-K,*)$, and $\hat{H}(\{*\}\times S^1)$ represents an element A (by abuse of notation) of $\pi_1(D,*)$ which is central. Since $\hat{M}$ is irreducible by [MSY],[D], D is Haken so by [W] D is Seifert fibered. Suppose $Z(\pi_1 M)$ is cyclic. T' bounds a solid torus, so the Seifert fiber space D is a cable space. Since A can be represented by a loop on either of T, T', the maps $H_1 S \to H_1\hat{M}$, $H_1 S' \to H_1\hat{M}$, where S' is the solid torus bounded by T', are onto. So $H_1 S\to H_1 S'$ is an isomorphism, and the cable space $D = T^2\times[0,1]$. So the end of $\hat{M}$ is of the form $T\times[0,\infty)$. T bounds a solid torus in $\hat{M}$, so $\hat{M}$ is $S^1\times\mathbf{R}^2$. Now suppose $Z(\pi_1 M)$ is infinitely generated. $H_1(E,\mathbf{Z}) \cong \mathbf{Z}\oplus Z(\pi_1 M)$, so, if L is large enough, $H_1(D,\mathbf{Z})$ is not generated by $H_1(T,\mathbf{Z})$. It follows that D has at least one multiple fibre, and therefore a hyperbolic base and unique Seifert fibering up to isotopy. There is a third torus T'' separating T' from ∞, such that the manifold D' with boundary components T', T'' admits a unique Seifert fibering, as does the manifold $D'' = D\cup D'$. Then the centralizer of $\pi_1 D'$ in $\pi_1 D''$ is $\langle\alpha\rangle$, where α is the fiber of the Seifert fibration of D'', and therefore can be represented by a simple closed curve on T. $A=\alpha^n$ for some n.

There exists $b \in Z(\pi_1 M)$, such that $a = b^m, m > n$, and corresponding homotopies $H(b) : M \times S^1 \to M$, $\hat{H}(b) : \hat{M} \times S^1 \to \hat{M}$. T' can be chosen far enough from T that some element B of $\pi_1 D'$ centralizes $\pi_1 D'$ in $\pi_1 E$ and $i_* B = b$, where $i : D' \to \hat{M}$ is the inclusion. By the uniqueness of the Seifert fibering, $B = \alpha^l$ in $\pi_1 D''$ for some l, $|l| \geq 1$. This is absurd: $i_* \alpha^{lm} = i_* B^m = b^m = a = i_* \alpha^n$, but $|l|m > n$. So $Z(\pi_1 M)$ is finitely generated, in fact cyclic. □

COROLLARY. *The center of any finitely generated 3-manifold group is finitely generated.* □

THEOREM 2. *Suppose that M is closed and aspherical, and $\pi_1 M$ has a normal cyclic subgroup $\langle a \rangle$ and an element b such that $[b]$ is of infinite order in $\pi_1 M/\langle a \rangle$. Then the covering space $M(a,b)$ of M corresponding to the subgroup $\langle a, b \rangle$ is an $\mathbf{R}$ bundle over a torus or Klein bottle.*

PROOF. By the missing boundary theorem [T] it suffices to show that some finite cover of $M(a,b)$ is standard. So we can assume that M is oriented and $\langle a \rangle$ is central. There is a cyclic homotopy H in M which lifts to $\hat{H}$ in $M(a,b)$ which has two ends. As in theorem 1, $M_{(a,b)}$ has an exhaustion by compact submanifolds with 2 boundary tori and therefore $M_{(a,b)}$ is homeomorphic to $S^1 \times S^1 \times \mathbf{R}$. □

Theorem 2 has also been proven in [HRS]. They don't assume $\langle a \rangle$ is normal. But by [S2,S3] this is the case unless $\langle a, b \rangle$ is represented by a singular torus in a characteristic submanifold, in which case the result is well known.

§2. *A bounded homotopy to a fibered structure*

We wish to choose the homeomorphism from $\hat{M}$ to $S^1 \times \mathbf{R}^2$ so that $\hat{H} : \hat{M} \times S^1 \to \hat{M}$ is uniformly close to the group action of S^1 on $\hat{M}$ determined by the homeomorphism.

PROPOSITION 3. *Let M be a closed irreducible aspherical 3-manifold with $Z(\pi_1 M) = \langle a \rangle$. Let $H : M \times S^1 \to M$ be a cyclic homotopy such that the tracks of H represent a, and let $\hat{H} : \hat{M} \times S^1 \to \hat{M}$ be a lift of H. There is a homotopy $J : \hat{M} \times S^1 \times [0,2] \to \hat{M}$ with compact support such that $J_0 = J|\hat{M} \times S^1 \times \{0\} = \hat{H}$ and there is a solid torus $B \subset \hat{M}$ such that, writing $J_2 = J|\hat{M} \times S^1 \times \{2\}$, $J_2|B \times S^1 \to B$ is a free S^1 action and $J_2((\hat{M} - B) \times S^1) \subseteq \hat{M} - B$. B will be referred to as a tube.*

PROOF. Let B_1 be an unknotted solid torus in $\hat{M}$ (i.e. $\hat{M} - B_1^0 \cong T^2 \times [0, \infty)$). Let B_2 and B_3 be larger unknotted solid tori such that: $B_1 \subset B_2 \subset B_3$, $d(\partial B_1, \partial B_2) > C$, $d(\partial B_2, \partial B_3) > C$, $B_2 - B_1^0 \cong T^2 \times [0,1]$, $B_3 - B_2^0 \cong T^2 \times [0,1]$. As in theorem 1, d is the distance function associated with some Riemannian metric pulled back from M and C is a bound on the lengths of tracks. There is a deformation retraction $D : (B_3 - B_1^0) \times [0,1] \to B_3 - B_1^0$ of $B_3 - B_1^0$ onto ∂B_2, such that $D((B_3 - B_2^0) \times [0,1]) \subset B_3 - B_2^0$ and $D((B_2 - B_1^0) \times [01]) \subset B_2 - B_1^0$. Extend D to $D :\ ((B_3 - B_1^0) \times [0,1]) \cup \hat{M} \times \{0\} \to \hat{M}$ so $D(p,0) = p$ for all $p \in \hat{M}$. Let E be an open collar neighbourhood of $\partial(B_3 - B_1^0)$ in $B_3 - B_1^0$ such that $d(\partial B_2, E) > C$. Let $f : \hat{M} \to [0,1]$ be a continuous function such that $f = 1$ on $(B_3 - B_1^0) - E$ and $f = 0$ on $\hat{M} - (B_3 - B_1^0)$. Define $J : \hat{M} \times S^1 \times [0,1] \to \hat{M}$ by $J(p, \Theta, t) = \ D(\hat{H}(p,\Theta),\ f(\hat{H}(p,\Theta))t)$. Then J is a homotopy with compact support (in fact given a point p in $\hat{M}$, J can be found fixing $(\hat{M} - N(8C,p)) \times S^1$.) Write $J_1 = J|\hat{M} \times S^1 \times \{1\}$; then $J_1(B_2 \times S^1) \subset B_2$ and $J_1((\hat{M} - B_2^0) \times S^1) \subset \hat{M} - B_2^0$. As a map of

pairs, $J_1 : (B_2, \partial B_2) \times S^1 \to (B_2, \partial B_2)$ is homotopic to a group action $J_2 : B_2 \times S^1 \to B_2$. By the homotopy extension property applied to the inclusion $\partial B_2 \subseteq \hat{M} - B_2^0$, J_2 extends to a map $J_2 : \hat{M} \times S^1 \to \hat{M}$ such that $J_2((\hat{M} - B_2^0) \times S^1 \subseteq \hat{M} - B_2^0$, and J_2 is obtained from j_1 by a homotopy supported in a neighbourhood of B_2. Now take B to be an S^1-invariant solid torus contained in B_2. □

PROPOSITION 4. *Under the hypotheses of proposition 3, if $G = \pi_1 M/\langle a\rangle$ is residually finite, M is a Seifert fiber space.*

PROOF. By [K] and [S1] it suffices to show that M has a finite cover which is a Seifert fiber space. Let F be a compact fundamental domain for G in $\hat{M}$ and let K be the support in $\hat{M}$ of the homotopy J given by proposition 3. There are finitely many translates $g_1 F, \dots, g_n F, g_i \in G$, such that $K \subseteq \operatorname{int} \cup_{i=1}^n g_i F$. There is an epimorphism $\phi : G \to H$ where H is finite and $\phi(g_1), \dots, \phi(g_n)$ are distinct. Let M' be the corresponding regular cover of M. The projection $\pi : \hat{M} \to M'$ embeds K and B. $M' - \pi(B)$ has a cyclic homotopy and therefore a cyclic normal subgroup. Since $M' - \pi(B)$ is Haken, it is Seifert fibered. It follows that M' is Seifert fibered. □

Fix a Riemannian metric h on M, and let h also denote the pull-back of h to $\hat{M}$. Let $d(,) : \hat{M} \times \hat{M} \to \mathbf{R}$ be the associated distance function.

THEOREM 5. *Given a closed irreducible aspherical 3-manifold M with $Z(\pi_1 M) = \langle a\rangle$ and corresponding cyclic homotopies $H : M \times S^1 \to M$, $\hat{H} : \hat{M} \times S^1 \to \hat{M}$, there is a homotopy $J : \hat{M} \times S^1 \times [0,3] \to \hat{M}$ such that $J_0 = J|\hat{M} \times S^1 \times \{0\} = \hat{H}$, $J_3 : \hat{M} \times S^1 \to \hat{M}$ is a free group action, and all tracks $J((p, \Theta) \times [0,3])$ have uniformly bounded diameter, i.e. J is a bounded homotopy.*

PROOF. Fix a basepoint $*$ in $\hat{M}$. There is a compact neighbourhood, say $K = N(10C, *)$ of $*$ and a homotopy of $\hat{H}$ supported in $K \times S^1$ to $\hat{H}_1$ which straightens the cyclic homotopy on a solid torus B containing $*$, by proposition 3. Take a subset N of G which is maximal with respect to $g_1 K \cap g_2 K = \varnothing$ for all $g_1 \neq g_2$ in N. Then $N^* = N \cdot *$ is uniformly dense in $\hat{M}$: for every $p \in \hat{M}, d(p, N^*) \leq C_2 = 20C$. There is a bounded homotopy, $J : \hat{M} \times S^1 \times [0,1] \to \hat{M}$, from $\hat{H} = J_0$ to $J_1 : \hat{M} \times S^1 \to \hat{M}$ where $J_1((N \cdot B) \times S^1) = N \cdot B, J_1((\hat{M} - N \cdot B) \times S^1) = \hat{M} - N \cdot B$, and $J_1 : N \cdot B \times S^1 \to N \cdot B$ is a group action. The bounded homotopy can be assumed to have support in $N \cdot K \times S^1$. We will now work in $\tilde{M} = \hat{M} - N \cdot B$. Having straightened out $\hat{H}$ on the "S^1-0-handles", we will straighten $\hat{H}$ on the "S^1-1-handles", a disjoint collection of properly embedded annuli of uniformly bounded diameter which divide $\tilde{M}$ into solid tori, the "S^1-2-handles", of uniformly bounded diameter, and finally straighten $\hat{H}$ on these. Let T be a collar neighbourhood of ∂B, small enough that $N \cdot T$ is a disjoint union. Replace h by h_1 where $h = h_1$ on $\tilde{M} - N \cdot B, h_1|gT = (g^{-1})^* h_1|T$ for all $g \in N$, and $\partial \tilde{M}$ is convex in the metric h_1. Write d_1 for the distance function of h_1, and $N_1(,)$ for metric neighbourhoods with respect to d_1. For each isotopy class of embedded essential annulus in $\tilde{M}$, there will be at least one embedded annulus of least area. If C_1, C_2 are boundary components of least area annuli A_1, A_2 and C_1, C_2 lie on the same tube, C_1 and C_2 are disjoint unless $A_1 = A_2$. Given two least area annuli A_3 and A_4, A_3 and A_4 intersect transversely in finitely many essential circles. These statements can also be established for PL least area annuli [JR]; in this case the triangulation is modified in T.

Fix a tube B_0. We will inductively construct trees $\Gamma(n)$ with vertices $B_0, \dots, B_n$ and edges $A_1, \dots, A_n$ where i) the B_i, $0 \leq i \leq n$ are distinct tubes, ii) one component of ∂A_i

is on ∂B_i and the other is on some ∂B_j, $0 \le j \le i-1$, iii) A_i minimizes area among all annuli joining a tube not equal to $B_0, \ldots B_{i-1}$ to one of the tubes $B_0 \ldots, B_{i-1}$.

LEMMA 1. *For each n, the union $\cup\Gamma(n) = B_0 \cup \ldots \cup B_n \cup \ \cup_{1 \le i \le n} A_i$ is embedded in $\tilde{M}$.*

PROOF. This is true for $n = 0$. Suppose it is true for $0 \le i \le n$. Let A_{n+1} be an annulus of least area among all annuli joining a new tube B_{n+1} to one of $B_0 \ldots B_n$. (Obviously A_{n+1} exists, and need not be unique.) Suppose that $\partial A_{n+1} \subseteq \partial B_{n+1} \cup \partial B_k$, and $A_{n+1} \cap A_j \neq \emptyset$, for some $i < j$. Distinct annuli, of least area in their relative homology classes, intersect either transversely in a single essential circle or not at all. Furthermore $k \neq i, j$. The proofs are simple trading arguments. Let $C = A_{n+1} \cap A_j$. Then there are annuli $A(\partial B_{n+1}, C))A(C, \partial B_k)$, $A(\partial B_i, C), A(C, \partial B_j)$ such that $A_{n+1} = A(\partial B_{n+1}, C) \cup_C A(C, \partial B_k)$ and $A_j = A(\partial B_i, C) \cup_C A(C, \partial B_j)$. Write a_n, $a(\partial B_{n+1}, C)$, etc. for the areas of $A_n, A(\partial B_{n+1}, C)$, etc. Then $a(\partial B_i, C), a(C, \partial B_j) > a(C, \partial B_k)$. Otherwise an annulus of area strictly less than a_{n+1} joining ∂B_{n+1} to one of ∂B_i, ∂B_j could be constructed by rounding the corners of (say) $A(\partial B_{n+1}, C) \cup_C A(C, \partial B_i)$. Suppose $k < j$. Then there is an annulus $A(C, \partial B_k) \cup_C A(C, \partial B_j)$ of area strictly less than a_j, which contradicts the choice of A_j. So $i < j < k$. But then the annulus $A(\partial B_1, C) \cup_C A(C, \partial B_k)$ has area strictly less than a_j, contradicting the choice of A_j. We are done by induction. □

Let $\Gamma(\omega) = \cup_{n \ge 0} \Gamma(n)$ be constructed inductively. $\Gamma(\omega)$ need not contain every tube, so we continue to construct trees $\Gamma(\beta)$ for countable ordinals β by transfinite induction: if β is a limit ordinal, and $\Gamma(\gamma)$ has been constructed for all $\gamma < \beta$, $\Gamma(\beta) = \cup_{\gamma < \beta} \Gamma(\gamma)$; if $\beta = \gamma + 1$ is a successor, and $\Gamma(\gamma)$ has been constructed but does not contain every tube, choose $\Gamma(\gamma + 1)$ to satisfy conditions i), ii), iii). There are only countably many tubes, so for some countable ordinal α, the tree $\Gamma(\alpha)$ contains every tube. Lemma 1 shows that no two annuli of $\Gamma(\alpha)$ intersect.

LEMMA 2. *i) The areas a_n are uniformly bounded above. ii) The annuli A_n have uniformly bounded diameter. iii) $\cup\Gamma(\alpha)$ is locally finite in $\tilde{M}$. (In i), ii), n ranges over all successor ordinals less than or equal to α).*

PROOF. i) $N \cdot B$ is uniformly dense, so for any n, $d(B_0 \cup \ldots \cup B_n,\ N \cdot B \backslash (B_0 \cup \ldots B_n)) \le 2C_2$. Using d_1, the inequality holds with a constant C_3. There are only finitely many equivalence classes of pairs (mB, lB) such that $d_1(mB, lB) \le C_3, (m, l \in N)$ under the equivalence relation $(gmB, glB) \sim (mB, lB)$ if $(gm, gn) \in N \times N$, because N is uniformly dense in G (with respect to the word metric associated with some fixed set of generators.) For each such pair there is a least area annulus A_{ml}; one of these has the biggest area C_4. Then $a_n \le C_4$ for all n. ii) Using PL minimal surfaces, i) gives a bound on the number of 1-simplices an annulus A_n meets. If A_n meets a 2-simplex Δ, A_n meets some 1-simplex of $\partial\Delta$. So the diameter of A_n is bounded, say by C_5. In a Riemannian setting, one may argue that $a_n \ge d_1(\partial_0 A_n, \partial_1 A_n) \cdot C_6$ where C_6 is a lower bound on the length of a closed curve in $\tilde{M}$ homotopic to a track of the cyclic homotopy. iii) This follows from ii): given a compact set K, an annulus which meets K has boundary components on two of the finitely many tubes which meet $N_1(C_5, K)$. □

Note that essentially the same proof shows that, in a complete Riemannian plane R with uniformly discrete, uniformly dense subset N there is an embedded tree, with vertices N and geodesic edges, and edge lengths are uniformly bounded.

Split $\tilde{M}$ on $\cup\Gamma(\alpha)$, obtaining a collection of components. Each component, say X has a connected boundary ∂X consisting of annuli from the A_β's ($\beta \le \alpha$ and not a limit

ordinal) and annuli which lie on tubes ∂B_β. The inclusion of any one of these annuli is a homotopy equivalence. $X = N_1(D, \partial X)$ for some D (say $C_3 + C_5$) independent of the component X. $N_X(,)$ means the metric neighbourhood with respect to the metric d_X obtained from h_1 by taking lengths of paths in X. Write $\partial X = \ldots A_n \cup_{C(2n)} T_n \cup_{C(2n+1)} A_{n+1} \ldots$ where each A_n is mapped to some $A_{\beta(n)}$ in $\tilde{M}$ by the natural map $X \to \tilde{M}$, each T_n is mapped to an annulus in some $B_{\gamma(n)}$, and each $C(n)$ is a circle. We say T_m, T_n have a combinatorial distance $d'_X(T_m, T_n) = |m - n|$. Adjoining additional annuli splits X into components. The concept of combinatorial distance extends in the obvious way to compact components.

LEMMA 3. *There are constants C_7, C_8, C_9 independent of the component X such that a) if ∂X is noncompact, there are annuli T_m, T_n in ∂X and an embedded annulus $A \subset X$ joining T_m and T_n such that i) diameter $A \le C_7$ and ii) $d'_X(T_m, T_n) \ge C_8$; and b) if ∂X is compact and has combinatorial length greater than C_9, then X contains an embedded annulus A joining two tubes T_m, T_n such that i) diameter $A \le C_7$ and ii) $d'_X(T_m, T_n) \ge C_8$.*

PROOF. a) Given two tubes T_k, T_l with $k < l$, choose $k' < k < l < l'$ so that $T_{k'}, T_{l'}$ are disjoint from $N_X(C_7, T_k)$ and $N_X(C_7, T_l)$ where the constant C_7 will be chosen. There are half-open annuli L_1, L_2 in ∂X, beginning at $T_{k'}, T_{l'}$. $N_X(2D, L_1) \cap N_X(2D, L_2) \neq \varnothing$. (Otherwise X would have two ends, and an open annulus, properly embedded in X running from one end to the other would separate $\tilde{M}$ into two unbounded components, each of which must meet $\cup\Gamma(\alpha)$. But $\cup\Gamma(\alpha)$ is connected and disjoint from the hypothetical annulus.) So there is an arc α of diameter $\le 4D$ joining L_1 and L_2. An embedded annulus A can be found in an arbitrarily small neighbourhood of $\hat{H}(\alpha)$. Choose $C_7 = 4D + C_6 + 1$.

b) Similarly, with E_1, L_1 and E_2, L_2 as indicated in Figure 1b, $N_X(2D, L_1) \cap N_X(2D, L_2) = \varnothing$ implies that X retracts onto the torus obtained by identifying E_1, E_2 to circles. But X is a solid torus if ∂X is compact. (X need not, however, be embedded by the natural map into $\tilde{M}$). Choose C_9 so large that it is possible to find $T_k, T_{k'}, T_{k''}, T_{k'''}, T_l, T_{l'}, T_{l''}, T_{l'''}$ such that $d_X(T_k, T_{k'}), d_X(T_{k''}, T_{k'''}), d_X(T_l, T_{l'}), d_X(T_{l''}, T_{l'''}) \ge C_{10}, d'_X(T_k, T_l), d'_X(T_{k'''}, T_{l'''}) \ge C_8$. C_{10} must be chosen so that at least one of the tubes between T_k and $T_{k'}$ is at a distance C_7 from T_k. This is possible because i) the tree $\Gamma(\alpha)$ has vertices of uniformly bounded valence ii) the new annuli which are added are distant from each other, so the maximum valence of the enlarged graph is also bounded. $\square$

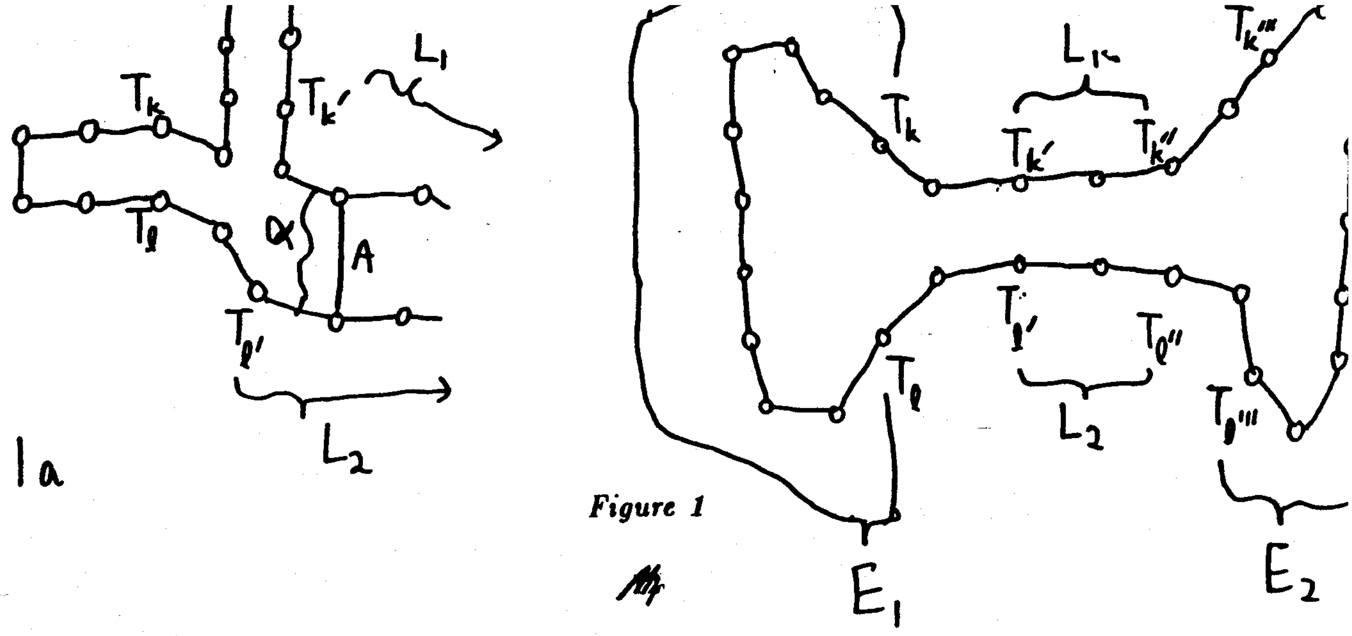

Figure 1

Figure 1

A collection $\{A_i\}_{i\in I}$ of annuli in X, such that

i) each annulus A_i is properly embedded, and its boundary components lie in annuli $T_{0(i)}, T_{1(i)}$,

ii) the A_i are disjoint, and

iii) each A_i joins two tubes at a combinatorial distance at least C_8

cuts X up into pieces of uniformly bounded size. Each A_i, $i \in I$ can be chosen to be area minimizing in its relative isotopy class; and moreover, the whole set of annuli $\{A_\beta : \beta \text{ is a successor ordinal and } \beta \leq \alpha\} \cup \{A_i : i \in I\} = \{A_i : i \in I_2\}$ can be chosen such that there are finitely many annuli say $A_1, \ldots A_N$ (after reindexing) such that $\{A_i : i \in I_2\} \subseteq G \cdot \{A_1, \ldots A_N\}$.

LEMMA 4. *There is a bounded homotopy $J : \tilde{M} \times S^1 \times [1,2]$ from J_1 to J_2, such that J is a homotopy relative to $N \cdot B$, and J_2 is an S^1 action on a regular neighbourhood $N(A_i)$ of each annulus $A_i, i \in I_2$.*

PROOF. Clear. □

LEMMA 5. *There is a bounded homotopy $J : \tilde{M} \times S^1 \times [2,3]$ from J_2 to J_3 relative to the tubes $N \cdot B$ and the neighbourhoods $N(A_i)$, so $J_{2.5}$ preserves each component of $\tilde{M} - \cup_{i\in I_2} N(A_i)$ and J_3 is an S^1 action.*

PROOF. In the covering space of $\tilde{M}$ corresponding to $i_*\pi_1 Y$ where Y is the closure of a component of $\tilde{M} - \cup_{i\in I_2} N(A_i)$, there is a deformation retraction onto Y. Argue as in proposition 3. □

This finishes the proof of theorem 5. □

It is convenient to introduce the concepts of quasiisometry, pseudoisometry and coarse quasiisometry. Let X and Y be metric spaces. $f : X \to Y$ is a C_1-quasiisometry if

$$C_1^{-1} d_X(x, x') \leq d_Y(f(x), f(x')) \leq C_1 d_X(x, x')$$

and a (C_1, C_2)-pseudoisometry if

$$C_1^{-1} d_X(x,x') - C_2 \le d_Y(f(x), f(x')) \le C_1 d_X(x,x') + C_2$$

($C_1 \ge 1, C_2 \ge 0$. Our (C_1, C_2)-pseudoisometry is a $(C_1+1, C_2(C_1^2+C_1))$-pseudoisometry using Mostow's definition [Mo].) A relation $R \subseteq X \times Y$ is a coarse quasiisometry if, for some C_1, C_2, M

a) if $(x,y), (x',y') \in R$, then

$$C_1^{-1} d_X(x,x') - C_2 \le d_Y(y,y') \le C_1 d_X(x,x') + C_2$$

and

$$C_1^{-1} d_Y(y,y') - C_2 \le d_X(x,x') \le C_1 d_Y(y,y') + C_2$$

b) Given any $x \in X$ there exists $x' \in X, d_X(x,x') \le M$, such that $(x',y) \in R$ for some $y \in Y$; and similarly for Y.

(Given a coarse quasiisometry R, it is always possible to increase C_2, decrease M to 0 and obtain a new, larger relation, $R \subseteq R'$.) See e.g. [G2],[Cp]. When possible we say f is a quasiisometry, etc. rather than a C_1-quasiisometry, etc.

COROLLARY 1. *G is coarse quasiisometric to some plane R, i.e. a complete Riemannian manifold homeomorphic to $\mathbb{R}^2$. We write this as $G \sim R$.*

PROOF. The lengths of the orbits of the S^1 action constructed in theorem 5 are uniformly bounded, in the original metric d. Average h and give the quotient space R the induced metric. □

COROLLARY 2. *R can be chosen to be quasihomogeneous, that is for every c there exists C and $r \in R$ such that every metric ball $B(c,p), p \in R$, is isometric to some $B(c,p')$ contained in $B(C,r)$.*

PROOF. There are only finitely many types of S^1-0-handles, S^1-1-handles, and S^1-2-handles up to the action of G on $\hat{M}$, and in each type of handle the S^1 action was constructed in the same way. (By way of justifying the terminology, quasicrystals are quasihomogeneous.) □

COROLLARY 3. *Suppose a group H is coarse quasiisometric to a complete Riemannian plane R. There exists a subset $N \subseteq H$ and an embedding of N in R which is a coarse quasiisometry, and an embedded geodesic 1-complex Γ in R, with vertex set N, uniformly bounded edge lengths, uniformly bounded valences of vertices, and complementary regions which are open 2-cells of uniformly bounded diameter.*

PROOF. It follows from the existence of the coarse quasiisometry $H \sim R$ that a uniformly dense subset $N \subseteq H$ embeds uniformly densely and uniformly discretely in R. The embedded 1-complex Γ is constructed by the argument in lemmas 2 and 3 of the proof of theorem 5. □

COROLLARY 4. *With the hypotheses of corollary 3, H is coarse quasiisometric to a quasihomogeneous plane. In fact there a triangulation $T(N')$ of R, with vertices N' of bounded valence, such that $H \supset N \subset N'$ and N is uniformly dense in H with respect to the word metric and in N' with respect to the (singular) Riemannian metric in which every triangle is equilateral.*

PROOF. The inclusion $N \subseteq R$ extends to a uniformly discrete quasiisometric embedding of H. Choose generators for H which include each $n_1^{-1} n_2$ where n_1, n_2 are endpoints of an edge of the geodesic 1-complex of corollary 3. Then the Cayley graph $\Gamma(H)$ of H (with respect to these generators) can be mapped into R, so that each edge is a distance

minimizing geodesic between its endpoints, extending the embedding of cor. 3. The image is uniformly locally finite. Any two edges intersect in at most 1 point. There is a bound on the number of edges of Γ an edge e of $\Gamma(H)$ crosses. Since there is a bound on the lengths of perimeters of complementary regions to Γ, the edge e can be replaced by a path in Γ, of some bounded length. It follows that $H \sim \Gamma$, where Γ is given the metric based on paths in Γ. To triangulate Γ, add extra vertices and extra edges in each complementary region. It is easy to see that the resulting $\Gamma(N')$ is quasiisometric to a union of equilateral triangles and, —adding curvature to the triangles near their vertices — to a (smooth) quasihomogeneous plane. $\square$

§3. *On groups which are coarse quasiisometric to planes*

Given a group G coarse quasiisometric to a plane R, which we may assume to be of bounded geometry, we use the quasiconformal structure of R to show that G is in fact coarse quasiisometric to a hyperbolic plane, if R is conformally a hyperbolic plane, and R is conformally equivalent to $\mathbf{C}$ if and only if G is amenable. By the uniformization theorem (e.g. [A2]) R is conformal to either $\mathbf{C}$ or $\mathbf{H}$.

THEOREM 6. *Suppose $N \subset G$ is C_1-dense, and a plane R of bounded geometry is triangulated with vertices N' such that $N \subset N'$ is C_2-dense in the metric on R and furthermore the relation $N \subset G \times R$ is a coarse quasiisometry: $C_3^{-1} d_G(n,n') \le d_R(n,n') \le C_3 d_G(n,n')$. Suppose (without loss of generality by Corollary 4 of theorem 5) that there are only finitely many isometry types of triangles in the triangulation $T(N')$. Then there exists C and $\phi_3 : G \times R \longrightarrow R$ such that*

i) *for each $g \in G$, $n \in N$, $d_G(\phi_3(g)n, gn) \le C$*
ii) *$d_G(\phi_3(gh)n,\ \phi_3(g) \circ \phi_3(h)n) \le C$*
iii) *each $\phi_3(g)$ is a diffeomorphism, and the diffeomorphisms $\phi_3(g)$ are uniformly bilipschitz:*

$$||D\phi_3(g)||,\ ||D\phi_3(g)^{-1}|| \le C$$

in particular, the diffeomorphisms $\phi_3(g)$ are uniformly C^2-quasiconformal).
iv) *and there is a subset $N'' \subset N$, uniformly dense in N, such that $\phi_3(g)N'' \subset N$.*

Sketch of Proof. G acts on G by translation and therefore on $N' \subset R$ by quasi-isometries, which extend to pseudoisometries $\phi_0 : G \times R \to R$ such that $\phi_0(gh)$ and $\phi_0(g) \circ \phi_0(h)$ are uniformly close. As in theorem 5, there is a bounded homotopy from ϕ_0 to $\phi_1 : G \times R \to R$, a uniformly quasiisometric set of homeomorphisms. Because R has bounded geometry, the set $\{\phi_1(g) : g \in G\}$ can be taken to be uniformly bilipschitz.

In the proof we use the convention that "$A \subset R$ is K-discrete" means that $d_R(a,b) \ge K$ for every $a \ne b$ in A.

PROOF.

LEMMA 1. *There exists $\phi : G \times N' \to N'$ such that*

i) *if $n \in N$, $d_G(\phi(g)n, gn) \le C_1$*
ii) *$d_G(\phi(gh)n,\ \phi(g) \circ \phi(h)n) \le 3C_1$ for $n \in N$*
iii) *if $m \in N'$ there exists $n \in N$ such that $d_R(m,n) \le C_2$ and for all $g \in G$, $\phi(g)m = \phi(g)n$*
iv) *if $m \in N', d_R(\phi(gh)m,\ \phi(g) \circ \phi(h)m) \le 3C_1C_3$*
v) *if $m, m' \in N'$ and $d_R(m,m') \ge 3(C_2 - C_3C_1)$ then*
a) *$d_G(\phi(g)m, \phi(g)m') \le 2C_1 + C_3(d_R(m,m') + 2C_2) \le C_4 d_R(m,m')$*
b) *$d_R(\phi(g)m, \phi(g)m') \le C_3^2 d_R(m,m') + 2C_1C_3 + 2C_3^2C_2 \le C_5 d_R(m,m')$*
c) *$d_R(\phi(g)m, \phi(g)m') \ge C_3^{-2} d_R(m,m') - 2C_3^{-2}C_2 - 2C_3^{-1}C_1 \ge C_5^{-1} d_R(m,m')$ for some constants C_4, C_5.*

PROOF. i) and iii) can be used to define ϕ, making arbitrary choices for each $m \in N'$. ii), iv), v) follow by the triangle inequality, for suitable C_4 and C_5. □

LEMMA 2. *There exists $\phi : G \times R \to R$, extending $\phi : G \times N' \to N'$ such that the maps $\phi(g)$ are (C_6, C_7)-pseudoisometric.*

PROOF. Extend the maps to the edges of the triangulation, mapping each edge n_1n_2 to a shortest geodesic between the endpoints $\phi(g)n_1, \phi(g)n_2$, and then to the 2-simplices of the triangulation. Because there are only finitely many possibilities up to isometry for

triples $(\phi(g)n_0, \phi(g)n_1, \phi(g)n_2)$ where $n_0, n_1, n_2 \in N'$ are the vertices of a 2-simplex, the singular 2-simplices can be chosen to have uniformly bounded diameter. $\square$

LEMMA 3. *Given $n \in N$, there exists a homotopy $\Phi : G \times R \times [0,1] \longrightarrow R$ and constants ϵ, C_8, C_9 independent of n such that i) $\Phi(g,r,0) = \varphi(g)r$ ii) $supp\Phi \subset G \times B(C_8, n)$, iii) tracks $\Phi((g,r)\times[0,1])$ have diameter $\leq C_8$, iv) each $\tilde{\varphi}(g)$ defined by $\tilde{\varphi}(g)r = \Phi(g,r,1)$ is a diffeomorphism from $B(\epsilon, n)$ to $B(\epsilon, \varphi(g)n)$ and $\tilde{\varphi}(g)^{-1}B(\epsilon,\varphi(g)n) = B(\epsilon, n)$ and v) the maps $\tilde{\phi}(g) : B(\epsilon, n) \to B(\epsilon, \phi(g)n)$ are C_{11}-bilipschitz.*

PROOF. There exist concentric circles S_1, S_2, S_3, S_4, S_5 bounding balls $B_1, \ldots B_5$ containing $\phi(g)n$ and a circle S_6 around n bounding a ball B_6 such that $\phi(g)B_6 \subset B_4$ and $\phi(g)(R - B_6^0) \subset R - B_2^0$. We sketch the required construction. Given vertices n_1, n_2 of the triangulation $T(N')$, let $d_T(n_1, n_2)$ be the number of edges in the shortest edgepath from n_1 to n_2. Let $N_T(m,n)$ consist of all 2 simplices Δ such that all vertices $v \in \partial\Delta$ satisfy $d_T(v,n) \leq m$. Let $\overline{N}_T(m,n) = N_T(m,n)\cup$ (all complementary domains with compact closure). Then $S_T(m,n) = \partial\overline{N}_T(m,n)$ is a circle. There is also a circle $S_T(m-1/2, n)$ which is the boundary of a regular neighbourhood of $\partial N_T(m,n)$ in $N_T(m,n)$ and separates $N_T(m-1,n)$ from $\partial N_T(m,n)$. Choose $S_1 = S_T(1, \phi(g)n), S_2 = S_T(1.5, \phi(g)n)$. Choose $S_6 = S_T(m_6, n)$ where the whole number m_6 is at least $C_6(1.5 + C_7)$. Let m_4 be the first whole number after $C_6 m_6 + C_7$, and let $S_4 = S_T(m_4, \phi(g)n)$, $S_5 = S_T(m_4 + 1, \phi(g)n)$. Choose S_3 to be a circle between S_2 and S_4. Let $f : R \to [0,1]$ be a continuous function such that $f = 0$ on B_1, $0 < f < 1$ on $B_2^0 - B_1$ and $B_5^0 - B_4, f = 1$ on $B_4 - B_2^0$, and $f = 0$ on $R - B_5^0$. Let $D_g : B_5 - B_1^0 \times [0,1] \to B_5 - B_1^0$ be a deformation retraction onto S_3 such that $D_g(B_4 - B_2^0 \times [0,1]) \subset B_4 - B_2^0$ and extend D_g to $R \times \{0\} \cup B_5 - B_1^0 \times [0,1]$ by $D_g(r,0) = r$. Then $R \times [0,1] \to R \times \{0\} \cup (B_5 - B_1^0) \times [0,1]$ defined by $(r,t) \mapsto (\phi(g)r, f(\phi(g)r)t)$ is continuous, and $H_1 : G \times R \times [0,1] \to R$ defined by $H_1(g,r,t) = D_g(\phi(g)r, f(\phi(g)r)t)$ is a homotopy. By a further homotopy $\phi(g)$ can be homotoped to $\tilde{\phi}(g)$ satisfying: a) $\tilde{\phi}(g) : B_6 \to B_3$ is a diffeomorphism in addition to b) $\tilde{\phi}(g)^{-1}B_3 = B_6$. b) was achieved by H_1. The support of Φ is in $G \times N_T(C_6 m_5 + C_7, n)$ (where $m_5 = m_4 + 1$). Similarly diameters of tracks can be bounded in d_T. As d_T is comparable to the metric on R, C_8 can be chosen independent of n, g. $\square$

Given a set $N'' \subset N$ of vertices such that $\varphi(g)|N''$ is injective, $\varphi(g^{-1})$ can be homotoped on small neighbourhoods of the points in $\varphi(g)N''$ to $\varphi^*(g^{-1}) : R \to R$ a (C_6, C_9) pseudoisometry where (using lemma 1 iv)) $C_9 = C_7 + 6C_1C_3 + 1$ and $\varphi^*(g^{-1})$ restricted to $B(\epsilon, \tilde{\varphi}(g)n)$ equals $\tilde{\varphi}(g)^{-1}$. Replacing C_7 by C_9 in lemma 3 leads to a constant C_{10} instead of C_8. (We can assume $C_{10} > C_8$.)

LEMMA 4. *Suppose $N'' \subset N$ is $(C_7 + 3C_6C_{10})$-discrete. Then there is a homotopy $H_2 : G \times R \times [0,1] \to R$ such that i) $H_2(g,r,0) = \varphi(g)r$, ii) each $\varphi_1(g) := H_2(g,r,1)$ is C_{11}-bilipschitz as a diffeomorphism $\varphi_1(g)|_{B(\epsilon,N'')} : B(\epsilon, N'') \to B(\epsilon, \varphi(g)N'')$, iii) the tracks of H_2 have diameter $\leq C_8$ so each $\varphi_1(g)$ is $(C_6, C_7 + 2C_8)$ pseudoisometric, and iv) there is a set $\{\varphi_2(g^{-1}) : R \to R, g \in G\}$ such that $\varphi_2(g^{-1})(R - B(\epsilon, \varphi(g)N'')) = R - B(\epsilon, N''), \varphi_2(g^{-1})^{-1}(R - B(\epsilon, N'')) = R - B(\epsilon, \phi(g)N'')$, and a) all maps $\varphi_2(g^{-1})$ are (C_6, C_{12})-pseudoisometric where $C_{12} = C_7 + 2C_{10}$—in fact each $\varphi_2(g^{-1})$ is a $(C_{10} + 6C_1C_3 + 1)$-bounded homotope of $\varphi(g^{-1})$; b) $d_R(\varphi_2(g^{-1}) \circ \varphi_1(g)x, x) \leq C_{10} + C_7 + C_6C_8 + d_R(\varphi(g^{-1}) \circ \varphi(g)x, x) \leq C_{13} = C_{10} + C_7 + C_6C_8 + C_1 + 2C_1C_3 + C_6^2C_1 + C_6C_7 + C_7$, c) $d_R(\varphi_1(g) \circ \varphi_2(g^{-1})x, x) \leq C_{14}$ for some C_{14}.*

PROOF. Because each $\varphi(g)(N'')$ is $3C_{10}$-discrete, for each g there is a homotopy of $\varphi(g^{-1})$, constructed by applying lemma 3 simultaneously to each vertex of $\varphi(g)N''$, so iv)

a) is satisfied. Then apply lemma 3 simultaneously to each vertex of N'', and the other statements follow. $\square$

LEMMA 5. *There is a constant* C_{15} *such that if* $f : R \to R$ *satisfies* $d_R(f(x), x) \leq max(C_{13}, C_{14})$ *then* f *is homotopic to the identity by a* C_{15}*-bounded homotopy.*

PROOF. We use the bounded geometry of R. First, by a bounded homotopy homotope f to f_1 so as to fix a uniformly dense, uniformly discrete set V_1 of vertices. After a second bounded homotopy f_2 fixes the edges of a cell decomposition with vertices V_1, and after a third f fixes every 2-cell. $\square$

Suppose N'' is chosen to be maximal with respect to being $C_{16} = \max(C_7 + 3C_6C_{10}, C_7 + C_6(2C_{15} + 1))$ discrete. Then for each $g \in G$, $\varphi_1(g) : R - B^0(\epsilon, N'') \to R - B^0(\epsilon, \varphi(g)N'')$ is a relative homotopy equivalence, and $\varphi_1(g) \circ \varphi_2(g^{-1})$ is bounded homotopic to the identity, in $R - B^0(\epsilon, N'')$ relative to the boundary. (If N'' were only $(C_7 + 3C_6C_{10})$-discrete, $\varphi_1(g) \circ \varphi_2(g^{-1})$ would be bounded homotopic to the identity in R, but not in $R - B^0(\epsilon, N'')$.)

Finally, $\varphi_1(g)$ is bounded homotopic to a diffeomorphism, by an argument like that of lemma 5. Alter the metrics of $R - B^0_\epsilon(N''), R - B^0_\epsilon(\varphi_1(g)N'')$ in a neighborhood of the boundary so that the boundary becomes convex. There is an embedded set $E(N'')$ of geodesic arcs in $R - B^0_\epsilon(N'')$, each shortest in its isotopy class, and of uniformly bounded length. The complementary regions are of bounded diameter, and by choosing the metric on $R - B^0_\epsilon(N'')$ to be quasihomogeneous, as in theorem 5, and suitably choosing the geodesic arcs, there will in fact be only finitely many isometry types of complementary regions. (When there is more than one shortest arc in an isotopy class, the shortest arcs are linearly ordered, as indicated in Figure 2. Always choose a leftmost or a rightmost arc, AB or CD in figure 2.)

Now we construct ϕ_3.

Map each edge $e \in E(N'')$ to a geodesic arc $\phi_3(g)e$ which is shortest in the homotopy class of $\phi_1(g)e$; because $\varphi_1(g)$ is a homotopy equivalence this arc is embedded—see e.g. [FHS]. Choose the image arc to be leftmost or rightmost. The graph $\tilde{E}(N'')$ is constructed from $E(N'')$ by removing ϵ-neighbourhoods of vertices and putting in a circle. $\varphi_3(g)$ can be chosen to map $\tilde{E}(N'')$ into R by $e \to \phi_3(g)e, B^0_\epsilon(N'') \to B^0_\epsilon(\varphi_3(g)N'')$. By discarding some vertices of N'' (e.g. a,b,c in Figure 3) it can be assumed that the closures of the regions complementary to $\tilde{E}(N'')$ are embedded 2-cells, as are their images. Extend ϕ_3 over the 2-cells. Because there are only finitely many isometry types of regions complementary to $\tilde{E}(N'')$, and only finitely many isometry types of regions complementary to $\phi_3(g)\tilde{E}(N'')$ where $g \in G$ is variable, ϕ_3 can be chosen to be bilipschitz, with a constant independent of g. (A region complementary to $\phi_3(g)\tilde{E}(N'')$ is determined, up to finitely many choices, by the sequence $(n_1, \ldots n_k)$, $n_i \in N$ such that the edges of the region join $B_\epsilon(n_1), \ldots B_\epsilon(n_k), B_\epsilon(n_1)$ in that cyclic order, and the knowledge of which points of N are in $\phi_3(g)N''$, in some region of size bounded independently of g.) $\square$

THEOREM 7. *Suppose the (finitely generated) group* G *is coarse quasiisometric to a complete Riemannian plane* R. *If* R *is conformally equivalent to* $\mathbf{H}$, G *is coarse quasi-isometric to* $\mathbf{H}$.

PROOF. Assume (without loss of generality by theorems 5 and 6), that R can be triangulated with vertices N', that the quasiisometry is given by inclusions $N \to G, N \to N' \subset R$ where N is uniformly dense in G and R, and $N'', \phi_3 : G \times R \to R$ satisfy the

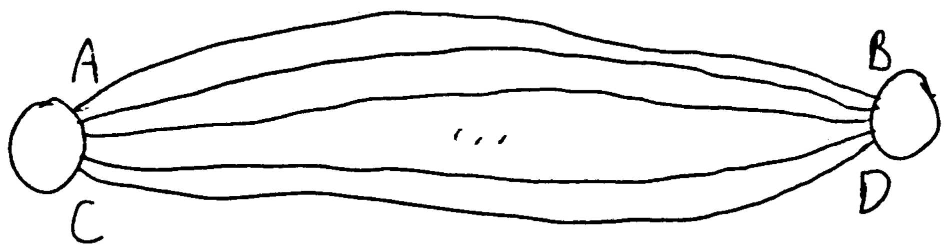

Figure 2

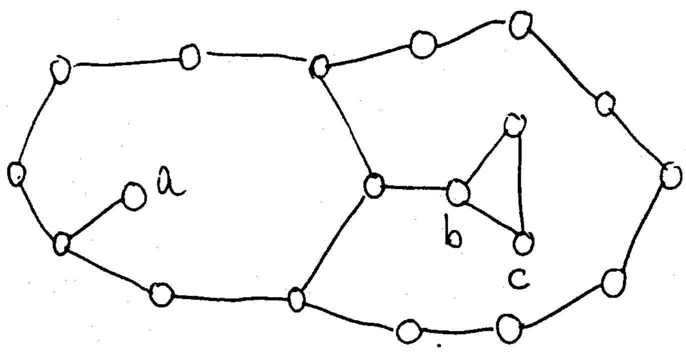

Figure 3

conclusion of theorem 6. Each map $\phi_3(g) : R \to R$ is C^2-quasiconformal. Given an edge e of $E(N'')$ joining $n_1, n_2 \in N''$ the endpoints $\phi_3(n_1^{-1})n_1$, $\phi_3(n_2^{-1})n_2$ are distinct and there are only finitely many possibilities for the pair $(\phi_3(n_1^{-1})n_1, \phi_3(n_1^{-1})n_2)$. Given two points r, s in R there is an open annulus $A(r,s)$ which separates $\{r, s\}$ from infinity in R and has maximal extremal width $d(A(r,s))$ by a standard application [St] of normal families. See e.g. [A1] for a discussion of extremal distance. Since C^2-quasiconformal maps change extremal distances by at most a factor C^2, the extremal widths $d(A(n_1, n_2))$ are bounded above and below:

$$C_{17} \leq d(A(n_1, n_2)) \leq C_{18}$$

It follows that the distance $d_H(n_1, n_2)$ in the hyperbolic metric is bounded:

$$C_{19} \leq d_H(n_1, n_2) \leq C_{20}$$

where $C_{19} = f(C_{17}), C_{20} = f(C_{18})$ and f can be expressed using elliptic modular functions as in [A1,A2]. Note that this does not say that N'' is $C_{19}/2$ discrete.

LEMMA 1. *$N'' \subset \mathbf{H}$ is C_{21}-dense for some constant C_{21}.*

PROOF. There is a bound k on the number of sides of the cells complementary to the geodesic network $E(N'')$. As in the proof of theorem 6, (see figure 3) we can assume that each of the complementary regions has embedded closure. Divide each cell into at most

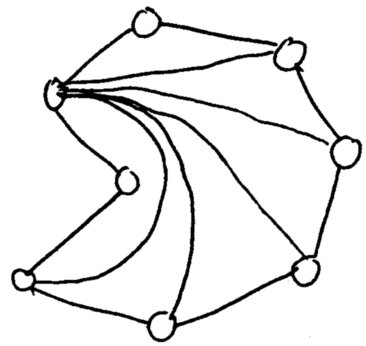

Figure 4

$k-2$ triangles, as in figure 4. This defines a locally finite 2-cycle on R which represents a generator of $H_2^{lf}(R,\mathbf{Z})$. Map this 2-cycle into $\mathbf{H}$, mapping each simplex to a geodesic simplex with the same vertices. The resulting locally finite 2-cycle covers $\mathbf{H}$, and edges have length $< k \cdot C_{20}$. So every point in $\mathbf{H}$ is within $C_{21} = k \cdot C_{20}$ of N''. $\square$

LEMMA 2. *Given m, n in N'', there is a chain $m = n_0, n_1 \dots n_p = n$ in N'' such that each pair (n_{i-1}, n_i) is the pair of endpoints of some edge e of N'', and $C_{22}^{-1} d_G(m,n) \le p \le C_{22} d_G(m,n)$.*

PROOF. The Cayley graph of G, the graph $E(N'')$ and the triangulation are coarsely equivalent to each other. $\square$

LEMMA 3. *Given $m \neq n$ in N'', $d_H(m,n) \le C_{20}C_{22} \cdot d_G(m,n)$.* $\square$

LEMMA 4. *Given a hyperbolic geodesic between m, n in N'', there is a broken geodesic $m = n_0, (n_0, n_1) \dots (n_{p-1}, n_p), n_p = n$ where (n_{i-1}, n_i) is a hyperbolic geodesic, $m, n_1, \dots n \in N''$, each $d_H(n_{i-1}, n_i) \le 3C_{21}$ and $\sum_1^p d_H(n_{i-1}, n_i) \le 3d_H(m,n)$.*

PROOF. Let $m = l_0, l_1, \dots l_p = n$ be equally spaced points on (m,n) with $d(l_{i-1}, l_i) \le C_{21}$. For each l_i choose an $n_i \in N''$ with $d_H(l_i, n_i) \le C_{21}$ and $n_0 = m, n_p = n$. Then $d_H(n_{i-1}, n_i) \le d_H(n_{i-1}, l_{i-1}) + d_H(l_{i-1}, l_i) + d_H(l_i, n_i) \le 3C_{21}$. $\square$

LEMMA 5. *There is a constant C_{23} such that if $d_H(n_{i-1}, n_i) \le 3C_{21}$ then $d_G(n_{i-1}, n_i) \le C_{23}$.*

PROOF. Apply $\phi_3(n_{i-1}^{-1})$ to the pair (n_{i-1}, n_i) obtaining g, h. There are finitely many possibilities for g : $d_G(e,g) \le C_1$. For any given g, $d_H(g,h) \le K(3C_{21}, C^2)$ by the compactness of normalized C^2-quasiconformal mappings. There are only finitely many

points of N in any compact set. So there are only finitely many possibilities for the pair (g,h). So $d_G(g,h)$ is bounded. Since $\phi_3(n_{i-1}^{-1})$ is a coarse quasiisometry with respect to d_G, $d_G(n_{i-1},n_i)$ is bounded. □

LEMMA 6. *There exists C_{24} such that if $m \neq n \in N''$ and $d_G(m,n) \leq C_{23}$, then $d_H(m,n) \geq C_{24}$.*

PROOF. As ususal there are only finitely many possibilities for the pair $(\phi_3(m^{-1})m, \phi_3(m^{-1})n)$. Because the maps $\phi_3(m^{-1})$ are uniformly quasiconformal, there is an upper bound on the extremal width of an annulus separating m,n from infinity and a lower bound on the hyperbolic distance from m to n. □

LEMMA 7. *$d_H(m,n) \leq C_{20}C_{22}d_G(m,n), d_G(m,n) \leq (3C_{23}/C_{24})d_H(m,n)$ for any m,n in N''.*

PROOF. Combine lemmas 3,4,5,6. □

It follows using lemma 1 that N'' is uniformly discrete (as well as uniformly dense) in $\mathbf{H}$, that N'' is coarse quasiisometric to $\mathbf{H}$, and that G is coarse quasiisometric to $\mathbf{H}$.

Here is an alternative argument, using lemmas 1-4. Since the maps $\phi_3(g) : g \in G$ are uniformly quasiconformal, they have uniformly quasisymmetric boundary values $\phi_\infty(g) : S^1_\infty \to S^1_\infty$. The maps $\phi_3(g)$ satisfy $d_H(\phi_3(gh)(n), (\phi_3(g)\circ\phi_3(h))(n)) \leq K_1$ for some K_1 and all n in N''. It follows that using lemma 1 $\phi_\infty(gh) = \phi_\infty(g)\circ\phi_\infty(h)$. The kernel of $\phi_\infty : G \to QS(S^1_\infty)$ is finite, because there is a bound K_2 such that a C^2-quasiconformal homeomorphism f which acts trivially on S^1_∞ satisfies $d_H(p, f(p)) \leq K_2$, and by lemma 4, if $\phi_\infty(g) = \mathrm{id}, d_G(e, \phi_3(g)\cdot e)$ is bounded.

Now $\phi_\infty(G)$ acts on $S^1_\infty \times S^1_\infty \times S^1_\infty - \Delta$ where $\Delta = \{(p,q,r) : p = q, q = r, \text{or } r = p\}$. Let us show that this action is properly discontinuous. $(S^1_\infty)^3 - \Delta$ has two components, one of which $[(S^1_\infty)^3 - \Delta]_+$ consists of triples in clockwise order. For given $(p,q,r) \in [(S^1_\infty)^3 - \Delta]_+$ and a neighbourhood $U \times V \times W$ of (p,q,r) (where $U \cap V, V \cap W, W \cap U$ are empty) let $x \in \mathbf{H}^2$ be the point on the geodesic pq such that the geodesic xr makes a clockwise 90^o angle with pq. By the compactness properties of quasisymmetric mappings, if $\phi_\infty(G)(U \times V \times W) \cap\ U \times V \times W \neq \phi$, $d_H(x, \phi_3(g)x) \leq K_3$ where K_3 depends on C^2 and the size of the neighbourhoods U, V, W. Again there are only finitely

many possible g's. $M = [(S^1_\infty)^3 - \Delta]_+/\phi_\infty(G)$ is compact. Otherwise M is a noncompact irreducible 3-manifold with finitely generated fundamental group and nontrivial center $Z(\pi_1 M) = \pi_1((S^1_\infty)^3 - \Delta, *)$. But then $\phi_\infty(g)$ is a free product of finite and infinite cyclic groups so G would have infinitely many ends contradicting the assumption that G is coarse quasiisometric to a plane. By theorem 8, a group G which acts uniformly quasisymmetrically on S^1_∞ such that $[(S^1_\infty)^3 - \Delta]_+/G$ is compact is coarse quasiisometric to $\mathbf{H}^2$. □

Remark 1. The hyperbolic plane admits metrics which are complete, have bounded geometry, yet satisfy $\lambda_0 = 0$ where λ_0 is the bottom of the spectrum of the laplacian. (By the well known theorem of [Br], if such a plane were coarse quasiisometric to a group, the group would be amenable. In [Br] it is assumed that the manifold $\tilde{M}$ such that $\lambda_0(\tilde{M}) = 0$ and $\tilde{M}$ is coarse quasiisometric to $\pi_1 M$ is actually the universal cover of M, but inspection of the proof shows that this is inessential.) This illustrates the strength of the hypothesis that R is coarse quasiisometric to G.

Remark 2. Suppose there is a closed 3-manifold M, $\mathbf{Z} \subset Z(\pi_1(M))$ and $G = (\pi_1 M)/\mathbf{Z}$. It can be shown, using theorem 3.1 of [HRS], that G is virtually euclidean, or a torsion group, or contains a free subgroup. If R is conformal to $\mathbf{C}$, then $\lambda_0(R) = \inf(f \text{ smooth with compact support})(\int_R df \wedge *df)\ /(\int_R |f|^2 dA) = 0$. To see this, suppose $\lambda_0 > 0$. By a theorem of Cheng and Yau [C-Y] it follows that there is a (unique, up to constants) positive harmonic solution to $\Delta u + \lambda_0 u = 0$. Then u is superharmonic. But $\mathbf{C}$ admits no positive superharmonic nonconstant function. I would like to thank S.Y. Cheng for this argument. So by [Br] and theorem 7, G is virtually euclidean, or torsion and amenable, or coarse quasiisometric to $\mathbf{H}^2$. This result will be superseded by theorem 10. However it would be of some interest to prove theorem 10 using quasiconformal geometry as in theorem 7 instead of random walks, and for that purpose the hypothesis "G is torsion" might be useful.

Theorem 8 is essentially due to Tukia [Tu2]; see also [Cp]. (In fact Tukia shows that the action is quasiconformally conjugate to a conformal action, if $n \geq 3$.)

THEOREM 8. *A (finitely generated) group G is coarse quasiisometric to hyperbolic*

space $\mathbf{H}^n$, $n \geq 2$, *if and only if* G *acts with finite kernel uniformly quasiconformally (uniformly quasisymmetrically, if* $n = 2$*) on* S^{n-1}_∞, *in such a way that* $((S^{n-1}_\infty)^3 - \Delta)/G$ *is compact.*

PROOF. Suppose an action of G on S^{n-1}_∞ is given. As in theorem 7, identify $(S^{n-1}_\infty)^3 - \Delta$ with the 2-frame bundle $V_2\mathbf{H}^n$. By the compactness of normalized quasiconformal (respectively quasisymmetric, if $n = 2$) mappings of S^{n-1}_∞, G acts quasiisometrically on $V_2\mathbf{H}^n$. It follows that $g \to \pi(g \cdot (p,q,r))$, where $\pi : V_2\mathbf{H}^n \to \mathbf{H}^n$ is the projection, and p,q,r are 3 fixed points, is a coarse quasiisometry from G to $\mathbf{H}^n$. Conversely, suppose $\phi : G \to \mathbf{H}^n$ is a coarse quasiisometry. Then G acts uniformly quasiconformally on S^{n-1}_∞, with finite kernel, and (assuming this kernel is empty) we obtain an embedding $G \to V_2\mathbf{H}^n$, $g \to g \cdot e$ where e is some fixed 2-frame. Given any 2-frame $f \in V_2\mathbf{H}^n$, there is some $g \in G$ such that $d_{H^n}(\pi(f), \phi(g)) \leq L$ where L is a defining constant for the coarse quasiisometry ϕ. Check that $g \cdot e$ is close to f. This shows that G is cocompact in V_2H^n. □

THEOREM 9. *Given two faithful actions* $G \overset{\Psi_1,\Psi_2}{\longrightarrow} QC(S^{n-1}_\infty)$ *(respectively* $G \longrightarrow QS(S^1_\infty)$*) each of which is cocompact on* $(S^{n-1}_\infty)^3 - \Delta$, *there is a canonical conjugating homeomorphism:* $h : \Psi_1 = h \circ \Psi_2 \circ h^{-1}$, *and* h *is quasisymmetric or quasiconformal, with a quality depending only on the qualities of* Ψ_1, Ψ_2.

PROOF. Theorem 8 provides coarse quasiisometries $\phi_1 : G \to \mathbf{H}^n, \phi_2 : G \to \mathbf{H}^n$. The relation $\phi_2 \circ \phi_1^{-1}$ is a coarse quasiisometry. h is the boundary value homeomorphism $(\phi_2 \circ \phi_1^{-1})_\infty$. □

Conjecture 1. Suppose G is coarse quasiisometric to $\mathbf{H}^2$. So there is a finite group $F \triangleleft G$ and $G/F \subset QS(S^1)$ is a uniformly quasisymmetric group, and there is a 3-manifold $M = [(S^1_\infty)^3 - \Delta]_+/G$ such that $Z(\pi_1 M) = Z \rightarrowtail \pi_1 M \twoheadrightarrow G$. Consider the 3 foliations P_1, P_2, P_3 of M such that each leaf of the preimage of P_1 in $[(S^1_\infty)^3 - \Delta]_+$ is of the form $L_p = \{(p,q,r) \in [(S^1_\infty)^3 - \Delta]_+ : q, r \in S^1_\infty - p, q \neq r\}$. M admits 3 Anosov flows X_1, X_2, X_3 such that: P_1 and P_2 are the expanding and contracting foliations for X_3, P_2 and P_3, respectively P_3 and P_1, are the expanding and contracting foliations for X_1, respectively X_2.

Observation 1. Suppose $G = \phi_\infty(G)$ is orientation preserving and quasiisometric to $\mathbf{H}^2$, and contains an element γ of order 3. Then with $\tilde{M} = [(S^1_\infty)^3 - \Delta]_+, M = \tilde{M}/G$, then order 3 symmetry $(p,q,r) \to (q,r,p)$ descends to M, and fixes a set $C_1, \ldots C_n$ of circles, one for each conjugacy class of cyclic subgroups of order 3. For instance $\{(p, \gamma p, \gamma^2 p) : p \in S^1\}$ projects to the circle corresponding to the conjugacy class of $\langle\gamma\rangle$. According to [Th] it follows that $C_1, \ldots C_n$ are fibers in a Seifert fibration, so G is a Fuchsian group.

Observation 1 is due mainly to Bob Edwards. It was also known to Rubinstein and Scott.

Observation 2. If $G = \phi_\infty(G)$, G is a rational Poincare duality group. If in addition G is torsion free, G is the fundamental group of a closed surface of negative Euler characteristic.

According to Rips's theorem ([G1]), G acts simplicially on a finite dimensional contractible polyhedron with finite stabilizers and compact quotient. From the spectral sequence of the group extension $Z \to \pi_1 M \to G$ it follows that $H^i(G, \mathbf{Q}G) = 0 (i \neq 2), 2(i = 2)$ and that if G is torsion free, $H^i(G, \mathbf{Z}G) = 0(i \neq 2), 2(i = 2)$. G is FL over $\mathbf{Q}$ and if G is torsion-free, G is FL over $\mathbf{Z}$. By theorem 10.1 in [Bw] G is a rational Poincaré duality group (and an integral Poincaré duality group, if G is torsion free), of formal dimension 2. By [Co] this implies that $b_1(G) > 0$, so M is Haken, hence Seifert fibered, so G is a surface group. Note that there is no need to quote [Mu], [E-M], [E-L]. Tukia has shown [Tu1] that if G is a discrete uniformly quasisymmetric group or more generally a convergence group acting on S^1, then either G has a "simple axis", i.e. there is an incompressible torus in $[(S^1_\infty)^3 - \Delta]_+/G$ or G contains a semitriangle group H as a finite index subgroup, i.e. H is generated by two elements a, b of orders p, q and ab is of order $r < \infty$. This extends a theorem of Nielsen as corrected by Zieschang in [Z] from virtually Fuchsian groups to convergence groups. In particular, Tukia proves, independently from [Co], that if G is a torsion free convergence group acting on S^1, G is a free group or the fundamental group of a closed hyperbolic surface.

In [G1], Gromov defines and studies hyperbolic groups. A hyperbolic group G has a boundary ∂G. It is easy to see that a hyperbolic group acts as a convergence group

on ∂G. On p. 112 of [G1], Gromov states that if G is a hyperbolic group and ∂G is homeomorphic to S^1, and G is torsion-free, then G is the fundamental group of a closed hyperbolic surface. (Gromov informs me that the hypothesis that G be torsion-free is intended on p. 112 of [G1].) Thus his result is confirmed by Tukia's work and by the argument above based on Rips's theorem.

If M is a 3-manifold with $Z(\pi_1 M) = \mathbf{Z}$, $\pi_1/Z(\pi_1)$ not a torsion group, and $G = \pi_1/Z(\pi_1)$ not a Euclidean crystallographic group, a boundary $\partial G = S^1$ on which G acts can be constructed. Sketch: Start with a collection of annuli in the covering space $M_{Z(\pi_1)}$, each of which covers a map $T^2 \to M$ and such that each maximal rank 2 abelian subgroup of $\pi_1 M$ corresponds to a unique annulus. A cyclic order can be defined on the collection of ends of the annuli. Adjoining Dedekind cuts gives an S^1, upon which G acts.

Scott (unpublished) has given a more metrical construction of the circle at infinity, like Floyd's construction [F] of the group completion, and can show that G acts as a convergence group on the circle at infinity. Using Tukia's result, this shows that G is a cocompact Fuchsian group (which implies that G is coarse quasiisometric to the hyperbolic plane), except possibly if G contains a finite index semitriangle group, in which case Tukia's result does not show that G is even coarse quasiisometric to the hyperbolic plane. Thus theorem 7 gives new information about i) semitriangle groups ii) the existence of elements of infinite order and iii) groups G not known a priori to arise as $\pi_1(M)/Z(\pi_1 M)$ for some 3-manifold M, as well as a different construction of an action on a circle at ∞ if $G = \pi_1(M)/Z(\pi_1 M)$). It seems possible that the proof that two dimensional Poincaré duality groups are surface groups can be simplified in this way.

§4. *Groups which are coarse quasiisometric to planes: random walks.*

THEOREM 10. *Given a closed P^2-irreducible 3-manifold M, suppose $\mathbf{Z} \triangleleft Z(\pi_1 M)$ and $G = \pi_1 M/\mathbf{Z}$ is coarse quasiisometric to R, a complete Riemannian plane of bounded geometry. Suppose R is conformal to $\mathbf{C}$. Then a) G is virtually rank 2 abelian b) M is Seifert fibered.*

PROOF. b) follows from a) by [S1]. Many readers will be unfamiliar with random walks on groups and their relationship with harmonic functions. [DS] is a very readable

introduction.

Step 1. Fix a disc D in R with smooth boundary ∂D. A flow to infinity in an n-dimensional Riemannian manifold from the boundary ∂D of a compact submanifold consists of a closed $(n-1)$-form φ (it is sometimes useful to consider the dual vector field $(*\varphi)^{\#}$), such that $\int_{\partial D} \varphi = 1$. φ has finite power if $\int_{M-D^{\circ}} |\varphi|^2 \ dV < \infty$. Claim: there is no flow to infinity from ∂D on R of finite power. Indeed if such a φ existed, then given any annulus A with $\partial_0 A = \partial D$, $\partial_1 A$ separating ∂D from infinity, there would exist a harmonic form Ψ on A with $\int_{\partial D} \Psi = 1$, $\int_A \Psi \wedge *\Psi \leq \int_A \varphi \wedge *\varphi \leq \int_{M-D^{\circ}} |\varphi|^2 dV < \infty$. But then the extremal widths of annuli A would be bounded, contradicting the fact that R is conformal to $\mathbf{C}$.

Step 2. There is no flow to infinity of finite power from a compact submanifold of $\hat{M}$ where $\hat{M}$ is provided with the metric, constructed in the proof of corollary 1 of theorem 5, in which S^1 acts isometrically on $\hat{M}$ with quotient R.

PROOF. Let $p : \hat{M} \to R$ be the projection. It is easy to see that the existence of a finite power flow to infinity is independent of the compact set, which may be taken to be $p^{-1}D$ for a disc $D \subseteq R$. Given a finite power flow φ, average φ using the S^1 action; the resulting flow $\bar{\varphi}$ is of finite power and projects to a flow on R.

Step 3. There is no flow to infinity of finite power from a compact set in $\hat{M}$ (with its metric pulled back from M) because a bounded change of metric changes the power by a bounded amount.

Step 4. Given a graph X with a set E of (undirected) edges, a (unit) flow on X is an $\mathbf{R}$ valued 1-chain φ satisfying $\partial\varphi = 1 \cdot P$ for some single vertex u. φ has finite power if $\Sigma_{e \in E} \varphi^2(e) < \infty$.

Claim: the Cayley graph of G has no unit flow of finite power. First, this is independent of the set of generators. In fact the existence of a finite power flow to infinity is a coarse quasiisometric invariant of a graph [DS]. Fix a Heegaard splitting of $M : M = H_1 \cup H_2$ where H_1 is a regular neighborhood of a bouquet of circles. Use these circles as generators of $G: \ G = \langle g_1^{\pm}, \dots g_k^{\pm} \rangle$. Now if the associated Cayley graph $X(G)$ admitted a finite power flow to infinity φ, $\hat{M}$ would admit a finite power flow to infinity say, Ψ, supported in $\hat{H}_1$

the preimage of H_1 and satisfying $\int_{D_e} \Psi = \varphi(e)$ where D_e is an (oriented) meridional disc of $\hat{H}_1$. Although this is obvious, here is a proof: There are k meridional discs $D_{g_1}, \ldots D_{g_k}$ in H_1 corresponding to the k generators of G. Cutting H_1 on $D_{g_1} \cup \ldots \cup D_{g_k}$ gives a manifold with corners, say B. ∂B contains ∂k discs $D_{g_1}, D_{g_1^{-1}}, \ldots D_{g_k}, D_{g_k^{-1}}$. Given a vertex v of $X(G)$, $\hat{H}_1$ contains a corresponding copy B_v of B. Edges $e_g(v), g \in S = \{g_i^{\pm 1} : 1 \leq k\}$ leave v : $e_g(v)$ joins v to vg. Let $E(v)$ denote the edges at v. For $v \neq u$, $\Sigma_{g \in S} \varphi(e_g(v)) = 0$. The restriction φ_v of φ to the edges $e_g(v)$, $g \in S$ can be written, by induction as $\varphi_v = \sum_{i=1}^{2k-1} \varphi_i$ where $\varphi_i = 0$ except on two edges say $e_{g(i)}(v), e_{g'(i)}(v), \varphi_i(e_{g(i)}) \geq 0$ and $\varphi_i(e_{g(i)}(v)) + \varphi_i(e_{g'(i)}(v)) = 0$. Now given a pair of faces, D_a, D_b of B where $a \neq b, a, b \in S$, there is a unit flow Ψ_{ab} from D_a to D_b, such that $\Psi_{ab}|_{N(D_a)}$ respectively $\Psi_{ab}|_{N(D_b)}$ is independent of b, respectively a, for some neighborhoods $N(D_a), N(D_b)$. Moreover Ψ_{ab} can be chosen to be 0 near ∂B. There is some C such that $\int_B |\Psi_{ab}| \cdot |\Psi_{cd}| dV \leq C$ independent of a, b, c, d. Now to the finite energy flow φ on $X(G)$ there corresponds a smooth flow Ψ from B_u to infinity on $\hat{M}$ such that, for $v \neq u$ $\Psi|_{B_v} = p^*(\sum_i \varphi_i(e_{g(i)}(v)) \Psi_{g(i), g'(i)})$ (Note that $g(i), g'(i)$ actually depend on v as well as i.) Let $a_i = \varphi(e_{g(i)}(v))$. Now

$$\int_{B_v} |\Psi|^2 dV \leq C(\sum_{i=1}^{2k-1} a_i)^2 \leq \frac{C}{4}(\sum_{e \in E(v)} |\varphi(e)|)^2$$

$$\leq \frac{C(2k-1)}{4} \sum_{e \in E(v)} |\varphi(e)|^2$$

so $\int_{\hat{M}} |\Psi|^2 dV \leq C' \sum_{e \in E(v)} |\varphi(e)|^2$.

Step 5. Given a graph X, and, for each vertex $x \in X$ a set $\{p(e) : e \in E(x)\}$ of nonnegative real numbers such that $\sum_{e \in E(x)} p(e) = 1$, random walk on X starting at a vertex x consists of a probability space Ω and random variables $\omega_n : \Omega \to X$, $n = 0, 1, 2, \ldots$ such that i) $\omega_0 = x$ for any $\alpha \in \Omega$, ii) $P(\omega_{n+1} = y | \omega_n = x) = \sum p(e)$ where the sum is over edges from x to y and iii) given ω_n, ω_{n+1} is independent of ω_i for $i < n$. More formally, $P(\omega_{n+1} = y | \omega_n = x) = P(\omega_{n+1} = y | A)$ for any subset A of the set $\omega_n = x$ which is in the σ-field generated by $\omega_1, \ldots \omega_n$. In particular, given $G = \langle g_1, \ldots g_1 | R \rangle$, random walk on $X(G)$ means the random walk such that $P(\omega_{n+1} =$

$g|\omega_n = h) = \frac{1}{2k} \cdot \#\{s \in S = \{g_1^{\pm}, \dots g_k^{\pm}\} : g^{-1}h = \varphi(s)\}$. Here $\varphi : \langle g_1, \dots g_n | \quad \rangle \to G$ is the canonical quotient map.

Random walk on $X(G)$ is either recurrent, that is random walk starting at $e \in G$ eventually returns to e with probability 1, or transient, that is, there is a probability $p > 0$ that the random walk escapes to infinity; and if random walk is recurrent, with probability one random walk returns infinitely often to e. Given a random walk, let $p_n(x, y)$ be the probability of being at y at time $m+n$ given that $\omega_m = x$. N, the number of times the random walk returns to e, is a random variable; $N = \sum_{n=1}^{\infty} 1_{(\omega_n = e)}$ and $E(N) = \sum_{n=1}^{\infty} P_n(e, e)$. Let p be the probability of eventually returning to e; then $E(N) = \sum_1^{\infty} p^n = p/(1-p)$ if $p < 1$, and the walk is recurrent if and only if $\sum_{n=1}^{\infty} P_n(e, e) = \infty$.

Now assume $G = \pi_1 M/\mathbf{Z}$. Since the Cayley graph of $X(G)$ admits no finite power unit flow to infinity, random walk on $X(G)$ is recurrent. See e.g. [DS],[LS] for the equivalence of the existence of a finite power flow and transience of random walk, and Theorem 1 of [V1] for an alternative approach. Recently Varopoulos proved [V2,V3,V4; see also M] the following theorem.

THEOREM A. *Suppose G is a finitely generated group and $\gamma(n)$ is the growth function of G (with respect to a generating set $S = S^{-1}$).*

a) If $n^A/\gamma(n) \to 0$ as $n \to \infty$, then given $\epsilon > 0$ $P_n(e, e) \leq Cn^{-A/2+\epsilon}$
b) If $\gamma(n) \leq Cn^A$ for some C, A and infinitely many n, then G is virtually nilpotent, and
c) for some nonnegative integer, $c_1 n^d \leq \gamma(n) \leq c_2 n^d$ and
d) $c_1 n^{-d/2} \leq P_n(e, e) \leq c_2 n^{-d/2}$
e) Thus random walk on $X(G)$ is transient unless G is virtually abelian of rank 0,1, or 2, in which case it is recurrent.

(Part b) is a slight strengthening, originally due to [VW], of Gromov's theorem on polynomial growth [G2]; c) is from [Ba].)

With G as the statement of theorem 10, G is virtually rank 2 abelian since G has 1 end. For this application only a) and b) of theorem A are required, as nilpotent 3-manifold groups are classified in [H].

THEOREM 11. *If a finitely generated group G is coarse quasiisometric to a Riemannian plane R, then G is coarse quasiisometric to the euclidean plane or to the hyperbolic plane.*

PROOF. By corollary 3 of theorem 5, we may assume R has bounded geometry. By theorem 7 we may assume R is conformally equivalent to $\mathbf{C}$. It is easy to embed the Cayley graph of G in $R \times S^1$ with bounded distortion and then step 4 applies; the rest of the proof of theorem 10 is unaffected. $\square$

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